Default Risk and Its Effect for a Bond Required Yield and Volatility*

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Abstract. The basic conclusion is that the default risk cannot be reflected completely in the volatility of the bond price and thus should be accounted separately. Hence, default risk can not be precisely assessed through option models and it should to be considered separately from the volatility risk. This point is proved analytically and supported by empirical results. Proposed measure for the default risk increment in the required yield is the old and good mathematical expectance of a loss at a default, which depend quasi linearly by credit spread. This measure of a default risk is analyzed and its effect for portfolio strategy is formalized in suppositions and propositions. Outcomes are in the necessity of the separate account for default risks increments while building portfolios of stocks and bonds, and for estimation of weighted average cost of capital (WACC). The latter leads to substantial increase of WACC when debt exceeds safe limit.

Аннотация. Основной вывод статьи состоит в том, что риск дефолта не может быть полностью отражен в волатильности цены облигаций. Следовательно, этот риск не может быть точно рассчитан через опционные модели и должен учитываться отдельно от риска волатильности. Эта точка зрения обоснована аналитически и подтверждена эмпирическими результатами. Для учета дополнительного риска в требуемой доходности предлагается классический подход — математическое ожидание потерь при дефолте, которое квазилинейно зависит от спрэда доходности. Эта мера риска дефолта проанализирована, и ее эффект для стратегии формирования портфеля формализован в гипотезах и выводах. Выводы заключаются в необходимости отдельного учета поправок на риск дефолта при построении портфеля акций и облигаций и для оценки WACC. Последнее приводит к существенному увеличению средневзвешенной стоимости капитала (WACC), когда долг превышает безопасный предел.

Key words: Risk of the bonds, required yield of the bonds, default risk, individual risk, common risk, the average price of the capital.

1. INTRODUCTION — THE PRICE VOLATILITY, DEFAULT RISK, DISCOUNT RATES AND THE RISK STRUCTURE OF RISKY BONDS

After CAPM and the connected theory of Modigliani-Miller emerged in the 1960-s, risk of securities (both stocks and bonds) became treated almost exclusively as a synonym for the price volatility. However these theories did not consider the possibility of default and, as a result, they could not include default risk. However, this approach was substantially reconsidered in the Merton's (1974) classical work. He stated that his option model completely considered default risk increments through volatility of the bond price. That would remove restrictions on default in CAPM and closely align with CARM Modigliani-Miller's theory[1]. At the same time, it is well-known that in reality cost of capital depends on debt to asset ratio (see e.g. Basel III framework[2]). Also it is known that Merton's option model allows to predict default for several months, not years[3]. And, describing risk as volatility implies stationarity assumption, which does not match with empirical evidences. So, there are two questions (at least). The first question — whether really default risk is

[1] It may be easily shown, that Modigliani-Miller's theorem may be proved with CAPM and using Hamada's formula.
[2] Basel III: A global regulatory framework for more resilient banks and banking systems [assessed 11.03.2014] //URL: www.bis.org/publ/bcbs189.htm
[3] Moody's KMV model: www.defaultrisk.com

* Риск дефолта и его влияние на требуемую доходность облигаций и волатильность.

completely reflected by volatility of bond price? The second question — at what extent required income of the bond may be calculated from the bond price volatility and at what extent (if any) it depends on the default risk?

From the fundamental assumption of investor's rational behavior it follows that market price for an asset depends on expected economic gains under estimated risk (see e.g. Damodaran, 2008a). For the security with fixed income and N years maturity, the investment value of security V is equal to its market value MV, if discount factor is equal to y=YTM (yield to maturity, which may be treated as the income rate, required by market), that is:

$$MV = PV\,(CF_{t,}\ y) = \sum_{t=1}^{N} \frac{CFt}{(1+y)^{t}} \qquad (1)$$

Here CF_t — fixed cash flows to the fixed income security.

In case of common stock there is an additional uncertainty (as the income is not fixed) and instead of fixed cash flows CF_t it is necessary to use expected cash flows E (CF_t) with an infinite time limit. The alternative is to set forecasting horizon, forecast E (CF_t) before this horizon, and then to use additional estimations for price at the horizon and its average grows rate. Thus, there is always an uncertainty caused by different ways to estimate E (CF_t), price at the horizon, and especially expected grows rate. If an income required by the investor, denoted as "k", does not coincide with market yield "y", then investor will probably get security price estimation different from the market value:

$$V = PV\,(CF_{t,}\ k) = \sum_{t=1}^{\infty} \frac{E(CFt)}{(1+k)^{t}} \qquad (2)$$

Thus, one investor's estimation of k may differ from market value simply because it may estimate risk by a way which is different from the one used by the majority of market participants. One may notice that any estimation of risk is inevitably subjective. The wide review of this problem was made e.g. by Damodaran (2008b). Clearly risk may be reflected either in required return k or in estimated grow rate g, using Gordon formula, where CF_0 is actual cash flow reported:

$$V = (CF_0\,(1 + g)) / (k - g) \qquad (3)$$

This may be useful for practical approach, but hardly gives a clear explanation of g and k together. With given discount rate k, and market price V, average g may be assessed, and *vice versa*.

The objective estimation of risks and expected cash flows can be reached only in the equilibrium, but the real markets are deprived of this convenient property underlying CAPM and APT. Fama and French (2006) examined whether the CAPM explains value premiums, and whether, in general, average returns compensate β in the way predicted by the CAPM. They stated that throughout 1926 to 2004 variation in β unrelated to size and the value-growth characteristic (which are the factors used in Fama and French model) goes unrewarded. In the real markets security prices are often distorted by irrational behavior of investors, or by market price bubbles, or by the market manipulations, despite all efforts of regulating authorities.

Besides the specified reasons, difference of market price from the investor's investment appraisal is the major condition for liquidity of the market. After all if one assumes that market price is completely fair and equal to the investor's appraisal, then, sale or purchase of any assets becomes senseless, taking into account commission fees. Under this assumption investors will not make any transactions just because this would incur losses. The only way to resolve this contradiction is to assume that investors make transactions, counting on change in market prices in future. But then it is obvious that an investment price of an asset worth for trade must be resulted by discounting with a rate different from market anticipation. From formulas (1) and (2) the source of volatility for market price is not clear. What is the main source of volatility — changes in expected cash flows, or changes in discount rates? Or, maybe, both? For stocks it would be reasonable to expect stronger dependence on cash flows, while for bonds, on the contrary, it would be reasonable to assume that as expected cash flows are fixed, so volatility of price is caused by the discount rates.

Moreover, as usually CAPM (and its updated versions, such as ICAPM) is applied for assessment of discount rates, then discount rates are considered as relatively stable (see, for example, Teplova and Shutova, 2011). Thus commonly cash flow volatility is considered responsible for volatility in stock prices. The first looks on assessing sources of volatility belongs probably to Campbell and Shiller (1988) who proposed model to investigate relation between dividend-price ratio and discount factors. Later Campbel and Shiller (1991) researched yield spreads and interest rate movements. And then much later Cochrane, and Piazzesi (2005, 2008) researched sources of bond risk premium and decomposed the yield curve. These results for portfolio theory and asset pricing were observed and cited in many works by Cochrane (2005, 2007).

Most recent results exposed in fundamental review by Cochrane (2011) are fairly convincing that, in contrast of the common view, volatility in discount rates mainly makes up volatility of market prices both for stocks and bonds. This conclusion, according to Cochrane (2011), looks absolutely unexpected. Of course, it is obvious that changes in estimation of financial risks by investors during crisis should lead to changes in prices. Surprise is that it attracts volatility in discount rates, instead of cash flows. And the most natural and most likely mechanism of the influence of risks to discount rates (required income) for bond should be probability of default. Same mechanism should operate for stocks as well, as in case of default owner of stock completely losing his investment. Besides, the prices of stocks and bonds depend often on similar factors of individual and common risk.

All known researches of fluctuations in market prices certainly show that both the bond and portfolio of bonds have common risk and the individual risks, and both are described by volatility. Most explicitly this effect emerged during the crisis of 2007, as described in the work of Baba, Naohiko, and Packer (2009). Worsening in a global market situation involves deterioration in financial position for wide set of corporations and, hence, increased default risks for bonds and stocks. Besides, increased are all the risks, that directly affect volatility in bond prices, such as interest rate risk, rating risk, liquidity risk, as described in the work of Brunnermeier, Markus (2009). And Froot, Kenneth A., and P. O'Connell (2010) discussed the reinsurance of intermediated risks.

The number of defaulting firms is a good measure for common risk, and as it is shown in Bruche and González-Aguado (2010), during recession the number of defaulted firms grows, and the average amount recovered on the bonds of defaulting firms tends to decrease, which authors interprete as the "credit cycle".

Tang and Yan (2010) identify implied volatility as the most significant determinant of default risk among firm-level characteristics. But actually they tested implication that default probability and credit spread increase with jump risk. Empirically, they measure jump risk using the slope of implied volatility over strike prices for S&P 500 index options and its dependence on GDP grows or fall. Their main hypothesis is: "Default probability and credit spread are lower if the GDP growth rate is higher, if the growth volatility is lower, if the consumer sentiment is stronger, and if the implied volatility smile of S&P 500 index options is flatter". So, this research did not investigate firm level characteristics and individual risk. Rather these results characterize common risk.

Thus, back to the central question: is default risk the major factor for volatility? Cochrane's (2011) results suggest it is not in long run. Then, next question is: can one get required income from volatility assessment? Let's remind that this assumption is one of the basic hypotheses underlying CAPM. Other important assumption for CAPM, and for the Modigliani-Miller's theory implicitly connected with CAPM, is the assumption of "ongoing concern". That assumption ignores default risk. Merton (1974) made nice try to eliminate this obvious lack of CAPM and Modigliani-Miller's theory. Change in the value of firm dV in Merton's work is described by Gauss-Winer stochastic process dz with volatility σ.

$$dV = (\alpha V - C)\, dt + \sigma V dz \tag{4}$$

In this formula α — income on the capital, and C — all payments to creditors and proprietors, both considered as known and determined. Stochastic process involved is only fluctuation of firm value σVdz as a result of stochastic change in market price. Thus, all the information on probability of a default in this model is perceived by participants of the market and indirectly reflected by them in change of σVdz. Here σ is instantaneous variance of the return on the firm per unit time. And "instantaneous" character of the σ is one of the most questionable point, added to the same points for α and C. Can "instantaneous variance" be calculated by 30, 90 or 120 days, or permanently?

However, while dz is Wiener-Gauss process with unit volatility, σ dz is not stationary process. And, as it is not stationary process, Black-Scholes model is not applicable. At the same time, if it was stationary process it would not be relevant to reality. This is difficult dilemma, but hardly Merton's solution for this problem is rigorous. Latest applications of this model made by Moody's KMV model or by e.g. Elizondo and Padilla (2008) use stationary processes, which is mathematically correct, but sometimes more difficult in coinciding with practice.

And back to the difference between "implied volatility" and actual volatility — while implied that volatility is constant, actual volatility is not. Use of "implied volatility" in option models may cause two types of biases. Firstly, using "implied volatility" for option price one may just find volatility which matches market price. For example, if one knows the market price V of a common share, and proper discount rate k, he may find from (3) "average pace of growth g" (or, *vice versa*, k from g and V). Does it mean that the firm will really grow with pace g? Secondly, as far as many investors use Black-Scholes model to calculate option price, the price for these options (like CDS) may be formed by "implied volatility", used by the majority of investors. One may suppose that CDS price will reflect default probability, because of the law of one price. But it is not necessary, as CDS is sometimes illiquid and CDS is not available for the most of corporate bonds. Fontana and Alessandro (2010) explored persistent negative CDS-bond basis during the financial crisis, and deviations from the law of one price were also researched by Garleanu and Pedersen (2011).

The probability of a default in Merton (1974) work turns out as a result of option model for artificial asset F (V, t) = V — Eq (V, t), that is a full firm value minus equity. Thus YTM (required income R (t)) is described by equation F (V, t) = B exp (-R (t) t), where B — face value of a debt. For time of maturity T, the award for risk R (t) — r_f is function of two indicators — volatility of firm operations (more precisely, its all-round price) σ and d = B exp (- r_f t)/V — relations resulted cost of face value of a debt at the moment of time t to firm all-round price during the initial moment of time.

Thus a situation when face value of a debt is less than market firm value is recognized as default. And the ratio of debt face value to full firm price is fairly perceived, as the characteristic of probability of a default. This point proved to be practical and holds in contemporary-styled models.

Another point that eventually holds is Merton's (1974) unequivocal identification concept of "risk" with volatility of bond price. This is required for alignment with CAPM and it implies assumption of direct dependence for required income of the "risk".

However this assumption does not hold an empirical check neither for stocks nor for bonds. It is well known, that for stocks it may hold at some extend, but for huge portfolios only. And portfolios under consideration should be different from market portfolio in CAPM sense. Other factors and other risks should be considered, as Merton (1973) described in his brilliant ICAPM theory, that has substantially improved CAPM, and probably got the multifactor models started. This theory now is widely applied for factor models, specified for different countries, as example for Japan by Tsuji (2011), and for theoretical researches by Cochrane (2008). The importance of different factors along with the value premium and the CAPM market portfolio was observed later in works by Fama and Kenneth (2006), Kenneth and O'Connell (2008). P. Maio (2012) discussed relation of multifactor models with the ICAPM, while neither ICAPM, nor other factor models count for default probability.

Coming back to questions that have been put in the beginning of section, both negative answers eventually emerge. Firstly, if investor holds bonds to maturity (on HTM — held to maturity basis), and as bonds cash flows to investor are fixed, then the main factor of risk is default risk. Secondly, dependence of required income from volatility and default risk should be various for bonds with a various rating and repayment terms. For the investor holding bonds as temporary asset (AFS, and MM — available for sale and marked to market), risks of liquidity and rating risk may be essential (e.g. see Cochrane, 2005). These risks undoubtedly should be reflected in the price volatility, as well as interest rate risk, currency risk and risk of inflation. But are these risks really essential? Perhaps, for long-term bonds mainly, as these risks linearly depends of maturity. Thus, it is logical to assume that dependence of YTM (yield to maturity) from volatility may be largely (while not completely) replaced with dependence of YTM on a duration.

At the same time, if the probability of a default fluctuates, following casual fluctuations of cash flows, then it may be reflected in price volatility. However, it is feasible only for the firm with high debt to asset ratio and insufficient level of operational cash flow. So, it should not be evident for the investment grade firm. Management of two factors of financial risk of corporation — debt to asset and cash flow is described by Zhukov (2012). Castréna, Déesa and Zaherb (2010) performed stress tests for corporate sector probabilities of default in euro area, under a wide range of domestic and global macro-financial shocks. The result is not a puzzle: shocks affect seriously default probability and bond return. While bonds returns at crisis were significantly affected, it is clear, that casual fluctuations of cash flows could not change essentially default risk for investment grade bonds if debt and asset quality is sufficient.

Relations between GDP, default risk and credit spreads were researched in above-cited work of Tang and Yan (2010). Exploring CDS spreads for S&P500 index, they find that average credit spreads decrease in GDP growth rate, but increase in GDP growth volatility and jump risk in the equity market. At the firm level, they state, that credit spreads generally rise with cash flow volatility and beta, with the effect of cash flow beta varying with market conditions. That seems to be true.

However, if one perceives increase in the general volatility, as a signal of a coming financial crisis, then it surely can be the indicator of increase in default risk. Thus, dependence of bond required income on volatility caused by common risk is not exposed to doubts; the issue is dependence on volatility caused by individual risk.

The main hypothesis is — an individual risk of default is irrelevant to volatility in bond price, and therefore it may not be obtained as option price with Black-Scholes model. To back up this hypothesis further, in sections 2 and 3 fairly simple theoretical model is suggested, and in 4 some of empirical results are added.

2. INFLUENCE OF DEFAULT PROBABILITY ON REQUIRED INCOME — THE EXPLICIT FORM

If one wants to completely separate default risk from volatility risk influence on required income of the bond, he should take an investor who is indifferent to price fluctuations. For this purpose investor who holds bond for maturity (HTM) is considered. Such investor will be indifferent to fluctuations in the bond price, given two factors — probability of a default p_d and a share of debt λ, which investor will lose in case of a default (maturity and interest). Thus, additional required income Δy demanded by the investor (to cover possible losses at the case of a default) will be expressed as function from these parameters, and this function should reflect sensitivity of the investor to risk exposure.

$$\Delta y = U(\lambda, p_d)$$

Despite simplicity of dependence for a bond spread Δy from p_d and λ, the author did not manage to find the correct one, so let's deduce it.

In particular, for this dependence Sharp, Alexander and Bailey (1999) proposed:

$$y = (y^* + \lambda p_d) / (1 - p_d) \qquad (5)$$

Here $y = k_d$ — required income for the bond with default probability p_d, λ — a part of the debt which will be lost at a default, $y^* = r_f$ required income of riskless bonds.

However, this expression contains an error, it is fair only at $\lambda = 1$ and demands correction — instead of $(1 - p_d)$ it should be $(1 - \lambda p_d)$ in (5). In the work of Denzler, Dacorogna, Müller, McNeil (2006), the wide review on probability of a default is described. And the similar formula deducted (but erroneously) for a spread of the bond, connected with probability of a default for a case of linear function of utility (neutral perception of risk).

Following the old and good classical approach by Morgenstern and von Neumann (1953), in the case of risk neutral investor, "risk-neutral equivalent" of expected return for a bill with face value F is:

$$E[F] = F(1 - p_d) + F(1 - \lambda) p_d = F(1 - \lambda p_d)$$

Further face value F, discounted on rate for risky bond $y = k_d$ is equal to riskless equivalent discounted on riskless rate r_f:

$$F / (1 + k_d) = E(F) / (1 + r_f) = F(1 - \lambda p_d) / (1 + r_f)$$

Expressing required income k_d through r_f and λ, it is easy to get for required income:

$$1 + k_d = (1 + r_f) / (1 - \lambda p_d) = 1 + (r_f + \lambda p_d) / (1 - \lambda p_d) \qquad (6)$$

Expression for a spread of required income of bonds is easy for receiving, subtracting from this expression riskless required income[4]:

$$\Delta y = k_d - r_f = (r_f + \lambda p_d) / (1 - \lambda p_d) - r_f = (r_f + \lambda p_d - r_f(1 - \lambda p_d)) / (1 - \lambda p_d) = (r_f + 1)\lambda p_d / (1 - \lambda p_d)$$

The result is linear dependence of spread from "risk" $p_d \lambda$:

$$\Delta y = k_d - r_f = (r_f + 1) p_d \lambda / (1 - p_d \lambda) \qquad (7)$$

Though this formula is deduced for 1 year, it is easy to extend to any T similarly (but corrected to the right) with above-cited work by Denzler, Dacorogna, Müller, McNeil (2006).

Taking probability of a default for 1 year is equal to $p_{d,}$ denote that probability for T years as q_d (in-

[4] Author begs pardon for so much elemental details, but as experience show, "devil is hiding in details".

cludes possibility of 1, 2, 3, 4 and 5 years default), and 1 — q_d will represent opposite event probability.

$$q_d = \sum_{t=1}^{T} p_d (1 - p_d)^{t-1}$$

If the compensation for investor doesn't depend on bankruptcy date (as debtor obligation comes into force at maturity T), then expression for T years is similar to (7), while q_d replaces p_d, and K_d, R_f are T years rates, so

$$K_d = (1 + k_d)^T - 1, R_f = (1 + r_f)^T - 1$$
$$\Delta Y = K_d - R_f = (r_f + 1)\, q_d\lambda / (1 - q_d\lambda) =$$
$$= (1 + k_d)^T - (1 + r_f)^T$$
$$\Delta y = k_d - r_f = [(r_f+1)\, q_d\lambda / (1- q_d\lambda) + (1+r_f)^T]^{1/T} \quad (8)$$

Separate estimation of λ and p_d (or q_d) is challenging problem. Both are unknown to the investor and both estimations always involve an element of subjectivity. In particular, in the cited work by Denzler, Dacorogna, Müller, McNeil (2006), the factor of losses λ was assessed as equal to 0.4 (as recovery rate was assessed at 0.6), proceeding from its average value, based on some statistical data. Hardly is it possible to consider this assumption as universal recipe. For each firm the estimations for λ and for p_d should be individual. And each of those (similarly to Merton's approach) may be represented in the form of synthetic security with option.

However, in case of investor neutral to risk, an independent estimation of λ and p_d is not required, as the estimation of default risk is known in the form of $w = p_d \lambda$, which is equal to a mathematical expectance of losses of the investor. And as this estimation depends linearly on spread, it can be getting directly under assumption that market estimate of risk is right. While the question remains: is this estimate of default risk true?

For a bond without imbedded options it may be obtained with formula (7) for 1 year or (8) for T years. But if the bond has imbedded options, OAS-spread instead of a G-spread should be used (assuming that value of options is correctly assessed).

Assuming that investor will be sensitive to a risk estimation, as mathematical expectance of losses at a default $p_d\lambda$ it is possible to describe this sensitivity by substitution $w=U(p_d\lambda)$ (or, in the most general case $w=U(p_d, \lambda)$), where nonlinear function U sets sensitivity of the investor to risk.

For example, one may use polynomial function:

$$U(w) = (r_f + 1)\, w^{1+b} / (1 - w^{1+b}) \quad (9)$$

where b > 0 reflects sensitivity level.

3. FORMALIZATION OF CONDITIONS AT WHICH REQUIRED INCOME CORRESPONDS TO A MATHEMATICAL EXPECTANCE OF LOSS

Despite of the intuitive evidence for correspondence between required income and mathematical expectance of loss, for theoretical justification it is necessary to formalize possible assumptions at which this dependence holds. Further the possible variant of formalization is proposed.

Supposition 1. The huge number of independent investors M operates in the bond market. And everyone can form any portfolio from N liquid risky bonds, where N is great enough. Additionally, riskless bonds are available with annual rate r_f. Besides, average required income N of liquid risky bonds makes $r_f + \delta$ at average default risk w = σ (these instantaneous variables may depend on time and some other parameters).

Remark 1. Presence of a considerable quantity of bonds with various risk and income rate is practical. Average excess of required income over riskless bonds is observed variable. While influence of the embedded options may change yield, it is possible to consider only a certain subset of a set of risky bonds, traded in market, which satisfies requirement.

Supposition 2. The instantaneous default risk for every risky bond does not depend on instantaneous default risk of any other bond, with some exceptions, which may be overcome by filtering out given list of bonds.

Remark 2. Independence of default probability for bonds is the simplifying assumption which is not always carried out. There are external factors (state of the economy or bank industry conditions or oil prices, etc.) which can change common risk of all securities simultaneously. However it can better be described as dependence of external factors, than as dependence of one bond from another. In the work of Blöchlinger (2011) the relation between rating and pricing of bonds was derived in a large market with a "quasi-factor" structure, constructed by author. Setting some conditions of *force majeure* through a vector of external parameters β, one may deduct that at the fixed value of a vector β default risks of different firms will be roundly independent. Exceptions are if an influence of one firm on others is unusually high. The latter can be observed in the exceptional case of the largest banks, monopolies, or other companies taking exclusive positions as suppliers, or buyers.

Supposition 3. The investor is capable to estimate instantaneous mathematical expectance of the loss at a default ($w=p_d\lambda$) for each bond, while the exact estimation of default probability and loss at a default separately may be inaccessible.

Remark 3. Ability of investors to estimate instantaneous default risk ($w=p_d\lambda$) is an assumption that may seem strong enough. However without that estimation any investment activity would be impossible. Besides, today many methods are developed for that kind of estimation.

Supposition 4. Sensitivity of the every investor for default risk ($w=p_d\lambda$) is described by a function U, that is monotonously increasing and continuous on an interval (0,1) and U (0) =0. If expected required income of investment instantaneously exceeds riskless rate above U (w), then investor buys bonds, and if it is below U (w), sells.

Remark 4. This assumption represents quite realistic approach to motivation for trade, though it ignores commissions and bid-ask spread (and it is possible to be improved on this way by adding it to U). While sensitivity of investors to risk can be various, it may be described in form (6). In the work by the author (Zhukov, 2013) an explicit form for default risk function was proposed, as polynomic function of risk factors. But it was not empirically checked.

Statement 1. If an average mathematical expectance of loss is σ for a great number of risky papers N1 <N, and it is below average additional income required δ (see Supposition 1) at big enough N1, for any sensitivity to risk U, an investor will prefer to exchange a portfolio of riskless bonds to an equally weighted portfolio of risky assets.

By supposition 1, in equally weighted portfolio of N1 risky bonds average risk decreases to the level below σ/N1 (portfolio volume is taken as 1). Further, owing to continuity U (w) by Supposition 4, at any sensitivity of investor to risk, at big enough N1 additional required income of a portfolio δ will exceed an individual estimation of risk U (σ/ N1) and, according to the Supposition 3, the investor will prefer risky papers.

Remark 5. Thus, the effect of risk diversification is observed not only for the volatility risk, but also for default risk defined as instantaneous mathematical expectance of loss. Also, what will depend on investor sensitivity to risk is only number N1. If this number is big enough for ALL investors, then ALL investors will change riskless portfolio to risky one if δ is greater 0, and therefore riskless rate must go up. While, taking in account common risk, investors may use CAPM to invest some money into riskless bonds, or use some hedging derivatives (like options for index) for the common risk hedging.

Statement 2. If sizes of M and N1 are big enough, then additional instantaneous income required δ1=Δy which investors demand for possible loss at a default of the one bond will tend to w=σ1 — instantaneous mathematical expectance of loss in the case of default. In other words, if quantities of securities N1and investors M are big enough, then investors become neutral to risk.

Proving this, the incremental risk of investor on the one bond at equally weighted portfolio is equal σ1/N1. As N1 can be as much as one wishes, risk may be reduced to size, when U (σ1/N1) <δ. Then, by Supposition 3 about a continuity of monotonous function U, investor will sell bond for which risk is above required income, replacing it with riskless one. Or, on the contrary, he will replace riskless bond with risky bond if risk is less than additional required income. Thus, bond price will go down until its risk exceeds income, or go up until income exceeds risk.

Remark 6. To some extent this may be supported by empirical data, see Gabaix, Xavier, Krishnamurthy, and Vigneron (2007) or Brunnermeier and Pedersen (2009). However this statement is still hypothetic and requires more empirical evidences.

Statement 3. Let's assess default risk and income for equally weighted bond portfolio, assuming that risk and income for each bond are random variables, dependent on time (previously risk and income were expected to be instantaneous). Denote by N1 number of bonds in and total portfolio risk by μ (expected loss in portfolio, dependent on time).

Then, while N1 tends to infinity, distribution of μ tends to normal. Hence, average portfolio income above riskless bond tends to constant value δ and average risk (mean of μ) tends to constant value σ. Besides variance of μ tends to 0 as square root of N1.

The proof clearly follows from Central Limit Theorem, as sum of independent stochastic variables tends to normal distribution (by Supposition 2 bond risks are independent), and volatility μ will tend to zero, proportionally square root from N1. However, it is clear, that average risk of risky bonds portfolio may not prevail over its average spread, as otherwise riskless bonds will be useless. Hence $\delta = (r_f + 1)\, \sigma / (1 - \sigma)$.

Let's draw some conclusions and assumptions.

1. It is possible to assume that though awards for risk of bond usually are not stationary in time

Table 1. Results for sample 1, BB rated bonds (76 bonds).

A. Results for full panel.

Call: lm (formula = ytm ~ mty + vol + oas + dta)

Residuals: Min	1Q	Median	3Q	Max
-1.5474	-0.8815	-0.5320	−0.1987	9.9402
Coefficients:	Estimate	Std. Error	t value	Pr (>|t|)
(Intercept)	1.8434050	0.8088247	2.279	0.0255 *
Mty	-0.4681503	0.3905977	−1.199	0.2345
Vol	-0.0040874	0.0353038	−0.116	0.9081
Oas	0.0086703	0.0015660	5.537	4.34e-07 ***
Dta	-0.0007702	0.0123905	-0.062	0.9506

Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 " . " 0.1 " " 1

Residual standard error: 2.16 on 75 degrees of freedom

Multiple R-squared: 0.3073, Adjusted R-squared: 0.2704

F-statistic: 8.319 on 4 and 75 DF, p-value: 1.311e-05

B. Results with parameters mty and oas excluded from panel.

Call: lm (formula = ytm ~ vol + dta)

Residuals: Min	1Q	Median	3Q	Max
-2.4913	-1.4511	-0.8886	0.3826	10.2238
Coefficients:	Estimate	Std. Error	t value	Pr (>|t|)
(Intercept)	3.542041	0.690647	5.129	2.12e-06 ***
Vol	-0.003744	0.041154	-0.091	0.928
Dta	-0.015229	0.014255	-1.068	0.289

Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 " . " 0.1 " " 1

Residual standard error: 2.542 on 77 degrees of freedom

Multiple R-squared: 0.01461, Adjusted R-squared: −0.01099

F-statistic: 0.5707 on 2 and 77 DF, p-value: 0.5675

(no less than required income of portfolio), but for the big and equally weighted portfolio under the fixed macroeconomic conditions and during the fixed time interval, expected loss and required income should be close to stationary. Probably, common risk is exactly the factor, which breaks stationarity, because of business cycles and crises. To get this effect parametrical dependence on common risk factors should be used.

2. In particular, assume (see Remark 2) that there are external factors of common risks β, reflecting business cycles, and shocks *(force majeur)* which can change risks of all securities simultaneously. Such parameters, in particular, may characterize general recession, or local recessions and lifting, and also crises of industries, technological shifts, bank crises, weather conditions, the prices of raw materials shocks and etc.

3. Notice also that the proposed account of default risk leads to changes in portfolio strategy. In particular, if the default risk makes primary impact on required income and risk of portfolio of bonds, it should be formed separately from a portfolio of stock, because principles of calculation of required

Table 2. Results for sample 2, AA rated bonds (329 bonds).

A. Results for full panel.

Call: lm (formula = ytmAA ~ dtaAA + oasAA + volAA + mtyAA)

Residuals: Min	1Q	Median	3Q	Max
-1.1076	-0.5593	-0.2849	0.0465	9.3262
Coefficients:	Estimate	Std. Error	t value	Pr (>|t|)
(Intercept)	0.064124	0.174745	0.367	0.713890
dtaAA	0.008257	0.002392	3.451	0.000631 ***
oasAA	0.010217	0.001347	7.587	3.45e-13 ***
volAA	-0.034001	0.013961	-2.435	0.015409 *
mtyAA	0.225331	0.113336	1.988	0.047629 *

Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 " ." 0.1 " " 1

Residual standard error: 1.19 on 326 degrees of freedom

Multiple R-squared: 0.222, Adjusted R-squared: 0.2125

F-statistic: 23.26 on 4 and 326 DF, p-value: < 2.2e-16

B. Results with parameters mty and oas excluded from panel

Call: lm (formula = ytmAA ~ dtaAA + volAA)

Residuals: Min	1Q	Median	3Q	Max
-4.4675	-0.6389	-0.2245	0.0810	9.7494
Coefficients:	Estimate	Std. Error	t value	Pr (>|t|)
(Intercept)	0.278544	0.166589	1.672	0.0955
dtaAA	0.010743	0.002612	4.112	4.96e-05 ***
volAA	-0.018948	0.015183	-1.248	0.2129

Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 " ." 0.1 " " 1

Residual standard error: 1.31 on 328 degrees of freedom

Multiple R-squared: 0.05088, Adjusted R-squared: 0.04509

F-statistic: 8.791 on 2 and 328 DF, p-value: 0.0001909

income of stocks and bonds essentially differ. At the same time, the risk of stock also needs revaluation taking into account default risk as it is not reflected completely in volatility risk.

4. The account of default risk leads to changes in WACC calculation. Usual assessment for opportunity cost of equity k_E may be represented with CAPM equality: $k_E = r_f + \beta_L$ MRP. Here $\beta_L = \beta_U$ (1 + D/Eq (1-T)) (Hamada equality).

Then: WACC = $k_{ps}w_{ps} + k_d w_d (1 — T) + k_E w_E$, where w_{ps}, w_d and w_E are shares of preferred stocks, debt and common stocks in capital, k_{ps} — interest on preferred stocks, k_d — average interest on debt. Taking into account probability of the default, required income k_d will no longer be a constant, it will depend on capital structure. It will lead to sharp increase in WACC at a growth of the debt over safe limits. For required income on equity one should use expression where Δy compensates default risk:

$$k_E = r_f + \Delta y + \beta L \text{ MRP} \quad (10)$$

If $k = r_f + \Delta y$ coincided with average interest for debt $k_{d,}$ it would not be a novelty. Many investment companies use this update for more realistic calculation of WACC. However, unlike the creditor who in case of a default can apply for a property part, the proprietor in case of a default theoretically loses all (that completely proves to be true in practice) as the default means that market value of assets is less than debt. Therefore for calculation of Δy in (10) it is necessary to accept λ =1, where $\Delta y= (r_f +1) p_d/ (1-p_d)$, which means that additional required income for default risk for stock is essentially above that for bond.

4. EMPIRICAL CHECK FOR DEPENDENCE OF BONDS REQUIRED INCOME FROM PARAMETERS OF VOLATILITY AND PROPERTY RISK

Empirical check of factors influencing the required income of bonds held two samples. Let's remember, that theory in section 3 used instantaneous values of required income and default risk. As trade is available at every moment, then it is supposed, that essential relations must be observable at any time, as information efficiency is observed.

So, the cross-sectional, one moment analysis was done for big number of bonds. All the data was taken from Bloomberg as of September 26, 2014. The statistical analysis has been done with use of R tools.

First sample consisted of 76 bonds rated BB (from BB- to BB +) with maturity from 1 to 2 years for which all necessary data were given in Bloomberg.

Second sample consisted of 329 bonds rated AA (from AA- to AA +) with the same maturity and data as the first sample.

Possible dependence (independence) of yield to maturity (ytm) from other variables was investigated. The independent variables were chosen in accordance with arguments exposed above (see Section 1 of this paper):

1. Debt to assets ratio (dta). Market value of assets is understood as a debt plus capitalization.
2. A spread for embedded options was taken into account (oas).
3. Duration was added as described above (mty).
4. 90-day volatility is added (vol).

The statistical data of results of research for 76 bonds of class BB (from BB- to BB +) is presented in Table 1, and for 327 bonds of class AA (from AA — to AA +) — in Table 2.

For the first sample (76 for bonds of class BB) required income does not depend on volatility with probability 0.9081, which completely confirms conclusions of Section 1. As default risk is highly essential to these bonds with low rating, then volatility risk does not render influence on required income.

The low correlation of YTM with OAS is a puzzle. Probably it means that option model used to calculate OAS doesn't work properly at moment, which again turn us to the question: how "implied volatility" of these models relates to actual volatility.

For the second sample (327 bonds of class AA) it is impossible to make such a definite conclusion — probability of independence is only 0.01099. And this fact also corresponds with conclusions in Section 1. For rated AA bonds the default risk during 2 years is rather low, and greater influence on their required income is rendered by factors of interest rate risk and common risk which are described by volatility. However, it is not so easy to explain why for the bond of classes AA and BB dependence on the debt to market ratio (dta) is absolutely different, namely: required income of bonds of class BB does not depend on this variable (with probability 0.9506), and required income of bonds of class AA, on the contrary, depends on it (probability of independence 1.418e-05).

It is a puzzle, as theoretically for BB that dependence should be higher, than for AA, as for BB probability of default is higher. The assumption is: dependence for bonds BB from debt to asset ratio is masked by dependence from OAS spread (oas).

To check it, parameters of a duration (mty) and OAS spread (oas) were excluded from panel and it caused some change in results — required income of class BB depends on debt to asset ratio, but significance is not small enough (0.289) while required income of class AA still depends on it with high probability (1–4.96e-05). So, puzzle still is not explained completely.

And another puzzle is — why dependence of YTM of OAS does not correspond with expected. The expectation was — correlation coefficient close to 1. But in reality it is at range 0.1–0.2, which means that option algorithms for OAS calculation doesn't align with practice. And, again, it is not against theory, as Black-Scholes model is not applicable for non-stationary process. But it is surprising, as option models are widely recognized as highly reliable. Perhaps, it may be caused by errors in volatility estimations. At the same time, the main purpose of empirical study was to check dependence of volatility. And independence YTM for BB bonds from 90-days volatility was not changed, which allows to assume stability and the importance of these results. Same is true for the low dependence of YTM for AA bonds from 90-days volatility, as this result sustained as well.

5. CONCLUSIONS

Let's repeat the basic conclusions.

1. Merton's conclusions that the default risk is completely reflected in volatility of the bond prices do not prove to be true by neither theoretical nor empirical results.

2. It is necessary to divide default risk which is understood as the mathematical expectance of loss at a default, and volatility risk which reflects common risks — percentage risk of bonds, risk of decrease in a rating, risk of liquidity and macroeconomic risks.

3. Required income on the bond essentially depends on default risk which may be treated as mathematical expectance of loss at a default. This dependence can be quasi-linearly expressed through a required income spread (7).

4. Though bond income usually is not stationary in time, but for the big and equally weighted portfolio under the fixed macroeconomic conditions and during the fixed time interval, expected loss and required income should be close to stationary. And, probably, common risk is exactly the factor, which breaks stationarity, because of business cycles and crises. To get this effect parametrical dependence on common risk factors should be used.

4. The external factors reflecting cyclical changes in the economy, and shocks caused by global or local crises *(force majeure)* which can change risks of all securities simultaneously, must be considered as external parameters. Change in these parameters eventually causes change in default risks and volatility risks.

5. Dependence of portfolio on default risk of separate bonds can be minimized at the expense of diversification. Thus criterion VAR of a portfolio for default risk of emitters becomes additive. At the same time, dependence on *force majeur* situations, the general for all emitters, should be considered as additional common risk.

6. Taking into account default risk, required income of a company debt and required income of its equity both should be increased, depending on capital structure. Thus, the account of default risk leads to changes in portfolio strategy and in WACC calculation. Required income of stocks and bonds should be increased by default risk. It will lead to increase for WACC at growth of a debt over safe limits.

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