

The Routes to Chaos in the Bitcoins Market*

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Abstract. I argue that the bitcoins market is an example of a complex system without a stable equilibrium. The users of bitcoins fall into two broad categories: 1) Capital gain seekers: who have no functional use for the currency apart from an expectation of capital gains; 2) Functional users: who use the currency to save on transaction costs as it provides a less costly medium of exchange over traditional fiat currencies. I assume that each category consists of mean-variance optimizers, and specify simple evolutionary dynamics for each category. I identify two simple routes to chaos in the bitcoins market. If only capital gain seekers are present, then one route to chaos is via the logistic map. If both categories of users matter then a possible route to chaos is via the delay logistic-Hénon map. A policy recommendation follows: in order to pre-empt chaos in the bitcoins market, currency exchanges should be allowed to convert bitcoins into dollars and vice versa if and only if there is an associated transaction involving buying and selling of goods or services or if the bitcoins are freshly mined. Such a regulation pre-empts chaos by reducing the impact of capital gain seekers on the virtual currency's value.

Аннотация. Рынок биткоинов является примером сложной системы, не имеющей устойчивого равновесия. Пользователи биткоинов делятся на две большие категории: 1) стремящиеся только к приросту капитала; 2) функциональные пользователи, которые используют валюту, чтобы сэкономить на транзакционных издержках. Я определяю два пути, которые могут привести к хаосу на рынке биткоинов. Если присутствуют только искатели прироста капитала, то маршрут к хаосу идет через логистическое отображение. Если обе категории пользователей присутствуют, то маршрут к хаосу возможен с помощью отсроченного логистического отображения Эно. Вывод: чтобы не допустить хаос на рынке биткоинов, валютным биржам следует разрешить конвертацию биткоинов в доллары и наоборот, только если имеется связанная сделка покупки или продажи товаров или услуг, или если биткоины только что получены. Такое регулирование предотвращает хаос путем уменьшения воздействия на стоимость виртуальной валюты со стороны пользователей, ищущих прироста капитала.

Key words: Bitcoins, chaos, speculation, digital currency, complex system, mean-variance optimization, medium of exchange, store of value, logistic map, delay logistic-Hénon map.

1. INTRODUCTION

Bitcoins is a decentralized virtual currency that emerged privately in 2009 with an objective of providing a low cost medium of exchange to facilitate transactions (locally as well as internationally). Financial institutions that facilitate transactions with government backed currencies need to invest heavily in infrastructure to be able to serve as effective facilitators of transactions. These infrastructural costs are passed on to merchants and customers in various forms such as credit card fees, wire transfer charges etc. The virtual currency bypasses this expensive infrastructure by utilizing the power of the bitcoin network based on peer to peer technology. Hence, it offers significant cost advantages as a medium of exchange. However, the virtual currency has no basis to

be a reasonably reliable store of value (as elaborated in Quiggin (2013), Krugman¹, DeLong², and others, it has no base-line intrinsic value as it cannot be used to pay off tax obligations).

The inability to be a reasonably reliable store of value has implications for the currency's effectiveness as a medium of exchange. An example clarifies: Suppose you are a merchant in the business of selling computers and you have been made aware of the significant cost advantages that would accrue to you if you start accepting bitcoins in exchange for computers. The system works as follows: When a customer buys a computer from you, you will send the bill to an online currency exchange. The online exchange will convert the total from dollars into bitcoins, and

¹ Bitcoin is Evil. Available at <http://krugman.blogs.nytimes.com>

² Watching Bitcoin, Dogecoin, Etc... Available at <http://equitablegrowth.org>.

* Пути к хаосу на рынке биткоинов.

a scanbar known as QR will be generated. The customer, upon receiving this request (via an app on his phone), will approve it and funds will move from his digital wallet to your digital wallet within seconds. On the positive side, you stand to save thousands of dollars per annum in credit card fees. However, on the negative side, your main concern is the following: How many dollars will I get when I convert bitcoins received through sales into dollars? To answer this question, you need to form an expectation about the exchange rate in the future when you intend to convert bitcoins into dollars. How do you even begin to form such an expectation? This challenge is reminiscent of Keynesian beauty contest in which one is required to form an expectation about what average opinion expects average opinion to be and so on.

It is useful to compare bitcoins with any other financial asset such as a share of a firm. There are two main differences: 1) A share of a firm is a claim on the earning stream generated by the activities of the firm. As a shareholder, your claim on the earnings generated by the firm is independent of whether you are a short-term investor interested in capital gains only or a long term investor interested in the dividend stream. In the case of shares, even if you are primarily a capital gain seeker, you collect dividends accruing to you while you hold the shares. In contrast, the benefits from bitcoins are directly dependent on how you intend to use them. If you are a *functional user*, like the merchant in the example above, you save on transaction costs. However, if you are a *capital gains seeker* with no functional use for the currency, then you do not benefit in terms of transaction costs. You are just hoping to buy it cheap and sell it expensive. 2) The benefit stream to functional users is directly dependent on how many functional users of the currency are there. Of course, if a lot of people are using bitcoins then more transactions will happen with the virtual currency, generating more savings for merchants accepting bitcoins. In contrast, the benefit stream to shareholders has no such direct dependence on the number of shareholders. The *real* earning stream associated with a given firm's productive activities is not directly dependent on the number of shareholders it has.

In this article, I argue that the differences pointed out above imply that the bitcoins market is best thought of as a complex system without a stable equilibrium. Like any complex system, the bitcoins market is also characterized by both the positive and the negative feedback mechanisms. There is positive feedback because greater the acceptability of the currency (as evidenced by the number of users), stronger is the incentive for outsiders to join in. There is negative feedback because a capital gain seeker wants to realize

profits before other capital gain seekers do. I show that the interaction of the positive and negative feedbacks generates routes to chaos. This paper adds to a large and growing literature on nonlinear dynamics and chaos in economics; see Orlando (2006) for a survey. An incomplete list of papers exploring the emergence of chaos in economic systems includes Brock and Sayer (1988), DeCoster and Mitchell (1991), Holyst (1996), Kopel (1997), and Kaas (1998) among many others.

Of course, expectations regarding the future value affect the values of all financial assets, however, for share price, there is an associated earning stream generated by the productive activities of the firm, which exists out there independent of the number of shareholders. In theory, it is possible to form objective expectations about the *real* earnings stream. Rational expectations finance postulates that it is possible to express such expectations in terms of known exogenous factors; hence, somewhat ironically, expectations do not matter in rational expectations equilibrium as they are anchored in objectively known exogenous factors. In contrast, in the bitcoins market, it is impossible to express expectations solely in terms of exogenously known factors as the associated benefit stream is not available to capital gains seekers, and even for functional users, who have access to it, the benefit stream is endogenously generated through mutual use. Clearly, one can see the signs of a complex system in which expectations collectively generate an outcome, which in turn causes these expectations to be revised, which changes the outcome and so on, without ever settling down, as there is nothing to settle down to. For a discussion on complex systems thinking in economics, see Arthur (2013).

This article is organized as follows. Section 2 models the bitcoins market in a mean-variance optimization framework and argues that the market is a complex system without a stable equilibrium. A complex system may display extremely rich dynamics including sensitive dependence to initial conditions or chaos. A system in chaos can take completely unrelated paths even when the initial conditions are arbitrarily close. Section 3 specifies evolutionary dynamics under the assumption that only capital gains seekers exist and shows that there is a route to chaos via the logistic map. Section 4 specifies evolutionary dynamics under the assumption that both functional users and capital gains seekers matter and shows that there is a route to chaos via the delay logistic-Hénon map. Section 5 considers the case when only functional users are present and shows that equilibrium with rational expectations can be considered. Section 6 concludes with a discussion of a policy recommendation to pre-empt chaos in the bitcoins market.

2. THE BITCOINS MARKET AS A COMPLEX SYSTEM

Suppose there are two types of users of the bitcoins: 1) Functional users who primarily use the currency to save on transaction costs. Of course, they remain exposed to capital losses/gains on the virtual currency; 2) Capital gains seekers who have no functional use for the currency apart from an expectation of capital gains. That is, they hope to buy it cheap and sell it expensive.

Assume that there are only two assets: 1) Bitcoins and 2) A risk-free asset that pays $(1 + r)$ for every dollar invested per period. That is, I assume that potential users of the currency can keep their wealth in dollars in which case, they earn the risk free rate, buy risky bitcoins, or hold any combination of bitcoins and dollars in accordance with their preferences. The wealth dynamics are given by:

$$W_{i(t+1)} = W_{it} (1 + r) + \check{R}_{i(t+1)} D_{it} \tag{1}$$

In (1), W_{it} is the total wealth of user i at time t . D_{it} is the number of units of bitcoins in possession of user i at time t . $\check{R}_{i(t+1)}$ is the random excess return over the risk-free asset per unit of the virtual currency accruing to investor i in one period. That is:

$$\check{R}_{i(t+1)} = \check{B}_{i(t+1)} + \check{P}_{t+1} - (1 + r) P_t \tag{2}$$

Where $\check{B}_{i(t+1)}$ is the monetary value of transactional advantage accruing to a functional user i of the virtual currency per period. Note, $\check{B}_{i(t+1)} = 0$, for capital gain seekers. That is, for capital gain seekers:

$$\check{R}_{i(t+1)} = \check{P}_{t+1} - (1 + r) P_t \tag{3}$$

Assuming all users are mean variance optimizers, the demand for bitcoins by user i can be obtained as follows:

$$\begin{aligned} \text{Max}_{D_{it}} \left\{ E_{it} [W_{i(t+1)}] - \frac{e}{2} V_{it} [W_{i(t+1)}] \right\} \\ \Rightarrow D_{it} = \frac{E_{it} [\check{R}_{i(t+1)}]}{e V_{it} [\check{R}_{i(t+1)}]} \end{aligned} \tag{4}$$

Where E_i and V_i are conditional expectation and variance operators respectively, and e is the risk aversion parameter. Equation (4) shows that the demand for bitcoins by user i is equal to the expected excess benefit of bitcoins over dollars, scaled down by risk aversion multiplied by conditional volatility.

The supply of bitcoins follows a known schedule as new bitcoins are generated and awarded to people called miners if they solve complex algorithms of increasing difficulty. I denote the total supply of bitcoins at time t by W_{it} .

The market price of bitcoins can be obtained by equating the total demand to total supply:

$$\sum_{f=1}^{F_t} D_{ft} + \sum_{c=1}^{C_t} D_{ct} = m_t \tag{5}$$

Where f and c are indices for functional users and capital gain seekers respectively. F_t is the total number of functional users at time t , and C_t is the total number of capital gain seekers at time t . F_t and C_t are evolving over time as more and more people are becoming aware of it with time. In the next two sections, I discuss two ways in which chaotic dynamics can arise in the evolution of F_t and C_t .

Define the total number of demanders at time t as follows: $N_t = C_t + F_t$. For simplicity, assume that the conditional volatility is the same for every user, and all users have the same coefficient of risk aversion.

From equation (5), it follows:

$$\frac{1}{N_t (1 + r)} \left\{ \sum_{f=1}^{F_t} E_f [\check{B}_{f(t+1)}] + \sum_{f=1}^{F_t} E_f [\check{P}_{t+1}] + \sum_{c=1}^{C_t} E_c [\check{P}_{t+1}] - m_t e V \right\} = P_t \tag{6}$$

In (6), each user forms an expectation about the price next period.

To appreciate the complexity of the situation, consider a functional users expectation about the price next period:

$$E_f \left[\frac{1}{N_{t+1}(1+r)} \left\{ \sum_{f=1}^{F_{t+1}} E_f [\tilde{B}_{f(t+2)}] + \sum_{f=1}^{F_{t+1}} E_f [\tilde{P}_{t+2}] + \sum_{c=1}^{C_{t+1}} E_c [\tilde{P}_{t+2}] - m_{t+1} eV \right\} \right] = E_f [P_{t+1}] \quad (7)$$

So, a functional user's expectation about the price next period depends on his expectation of other users' expectations about the price in the next to next period. Trying to find out the expectation about the price in the next to next period leads to expectations about expectations about expectation. Clearly, one can continue this to obtain an infinite regress of expectations about expectations. This issue in financial markets led Keynes (1936) to form an analogy with a fictional newspaper beauty contest in which the contestants are asked to choose from among the set of six photographs of women. Those who choose the most popular one are declared winners.

"It is not a case of choosing those [faces] that, to the best of one's judgment, are really the prettiest, nor even those that average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees." (Keynes, *General Theory of Employment Interest and Money*, 1936).

Rational expectations finance found a way to work around this problem by postulating that agents hold (rational) expectations which are consistent with observable outcomes. This requirement imposes a very strong condition on allowable expectations as it eliminates expectational heterogeneity. In theory, one can make a case for rational expectations when it comes to share prices as shares are claims on earning streams that exist largely independent of shareholders. As earning streams of firms have a reality independent of shareholders, wrong expectations can be corrected through negative feedback, and in theory, a stable equilibrium may exist. However, there is difficulty in accepting this line of thinking as one clearly sees heterogeneous expectations in financial markets.

If the idea of rational expectations is hard to swallow for the relatively stable stock markets, this notion seems largely irrelevant in the bitcoins market. In the bitcoins market, the benefit stream, which is only available to functional users, not only varies from person to person, but is directly dependent on the number of functional users. That is, the benefit stream is endogenously generated. Even if we impose expectational homogeneity on functional users, there are capital gain seekers who do not get any functional benefits. They are simply hoping to buy cheap and sell expensive.

It is useful to think of the bitcoins market a complex system with both positive and negative feedbacks. Existence of both kinds of feedbacks is the defining feature of complex systems. See Arthur (2013) for a discussion on this kind of thinking in economics.

There is positive feedback for two reasons: 1) Greater the number of functional users, higher is the expected functional benefit, so addition of more functional users creates incentives for more functional users to join in. 2) Greater the number of users of the virtual currency, faster is the word of mouth influence, which has been the best advertisement for the currency.

There is negative feedback for the following reasons: 1) Greater the number of capital gain seekers, higher is the chance that profit taking would depress the price next period. 2) Functional users are also aware of the possibility of profit taking by capital gain seekers; hence, they may also adjust their demand downwards in anticipation of a decline.

In the next section, I discuss the implications of the positive and negative feedbacks for price dynamics in the bitcoins market.

3. BITCOINS MARKET: THE SIMPLE CASE

It is useful to discuss the simple case first, in which, there are no functional users. That is, only capital gain seekers are present. The price dynamics are then described by the following equation:

$$\frac{1}{C_t(1+r)} \left\{ \sum_{c=1}^{C_t} E_c [\tilde{P}_{t+1}] - m_t eV \right\} = P_t \quad (8)$$

To simplify matters further, I assume that short selling is not allowed. A capital gain seeker has a positive demand for bitcoins at price P_t if the following holds:

$$E_c[\tilde{P}_{t+1}] > P_t(1+r) \tag{9}$$

Note, that inequality (9) holds for all capital gain seekers who invest in bitcoins at time t . It follows:

$$E_c[\tilde{P}_{t+1}] = P_t(1+r) + f_c(P_t) \tag{10}$$

Where f_c is individual specific and a decreasing function of P_t . At the market clearing price, $f_c(P_t)$ is zero for the marginal capital gain seeker.

Substituting (10) in (8):

$$\frac{1}{C_t(1+r)} \left\{ C_t P_t(1+r) + \sum_{c=1}^{C_t} f_c(P_t) - m_t eV \right\} = P_t \tag{11}$$

$$\Rightarrow \sum_{c=1}^{C_t} f_c(P_t) = m_t eV \tag{12}$$

It is clear from equation (12) that the dynamics of C_t are the key to understanding the price dynamics in the bitcoins market.

The bitcoins market can be thought of as a complex system in which both positive and negative feedbacks influence C_t . As awareness of bitcoins is spreading through word of mouth, one may consider the following for positive feedback: aC_{t-1} where a is a positive constant, and C_{t-1} is the number of capital gain seekers with positive demand at time $t-1$. There is also negative feedback, as the sole objective of a capital gain seeker is to realize capital gains before the bubble bursts. So, the presence of other capital gain seekers also makes a given capital gain seeker wary as he wants to realize capital gains before they decide to do so, as that would depress the price. In short, greater the number of capital gain seekers, bigger is the bubble. The bigger the bubble, stronger is the chance of it bursting in the next period. I postulate that the probability that a given capital gain seeker would realize capital gains by selling is proportional to the total number of capital gain seekers in the market at a given point in time. Introducing a constant of proportionality b , one may reason that the probability that a given capital gain seeker would realize profits is: bC_{t-1} . Hence, the expected number of users who realize capital gains is: bC_{t-1}^2 . Combining the positive and negative feedbacks, one may write:

$$C_t = aC_{t-1} - bC_{t-1}^2 \tag{13}$$

One can study the qualitative dynamics for various values of the parameters.

Substitute (13) in (12):

$$\Rightarrow \sum_{c=1}^{aC_{t-1} - bC_{t-1}^2} f_c(P_t) = m_t eV \tag{14}$$

It is clear from equation (14) that the dynamics of (13) are the key to understanding the dynamics of (14). (13) can be transformed into the famous logistic equation with a simple change of variables as follows:

$$C_t = aC_{t-1} \left(1 - \frac{b}{a} C_{t-1} \right)$$

$$\text{Define } x_{t-1} = \frac{b}{a} C_{t-1}$$

$$x_t = ax_{t-1}(1 - x_{t-1}) \tag{15}$$

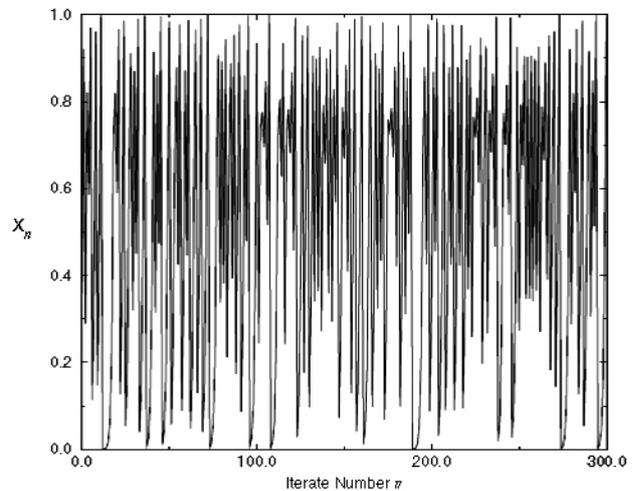


Figure 1. The behavior of logistic map when $a = 4$.

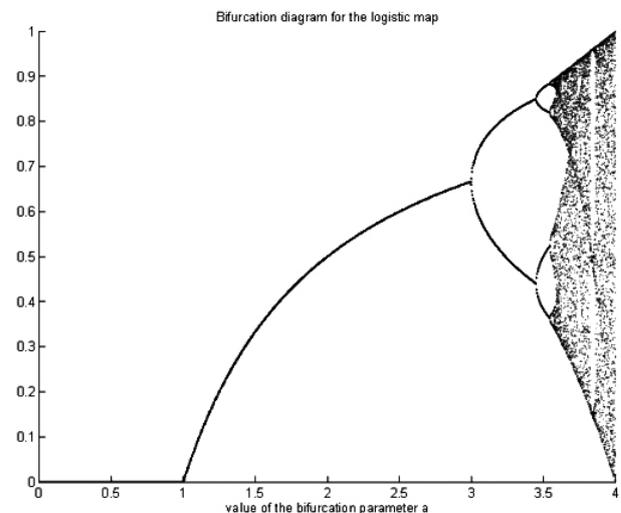


Figure 2. As Figure 2 shows, the logistic map has interesting dynamics including emergence of chaos beyond a certain threshold (approximately beyond $a = 3.8$).

(15) is the famous logistic map, which has been extensively studied in nonlinear dynamics and chaos literature over the past 25 years. A recent reference is Groff (2013). chaos, or sensitive dependence on initial conditions, arises in the logistic map if the value of exceeds a certain threshold. Figure 1 shows one path when $a = 4$.

The behavior of the logistic map can be summarized for various values of 'a' as a bifurcation diagram with a period doubling route to chaos as Figure 2 shows.

4. BITCOINS MARKET: THE GENERAL CASE

If both the functional users and the capital gain seekers are present, then the price dynamics are described by (6) which is re-produced below:

$$\frac{1}{N_t(1+r)} \left\{ \sum_{f=1}^{F_t} E_f [\tilde{B}_{f(t+1)}] + \sum_{f=1}^{F_t} E_f [\tilde{P}_{t+1}] + \sum_{c=1}^{C_t} E_c [\tilde{P}_{t+1}] - m_t eV \right\} = P_t \tag{16}$$

For simplicity, and without loss of generality, I assume that all functional users form identical expectations. That is:

$$\frac{1}{N_t(1+r)} \left\{ F_t (E_f [\tilde{B}_{t+1}] + E_f [\tilde{P}_{t+1}]) + \sum_{c=1}^{C_t} E_c [\tilde{P}_{t+1}] - m_t eV \right\} = P_t \tag{17}$$

Next, I specify the dynamics of F_t and C_t .

The dynamics of C_t are identical to the one described in the previous section except that now the word of mouth influence has been supplemented due to the presence of functional users:

$$C_t = a(C_{t-1} + F_{t-1}) - bC_{t-1}^2 \tag{18}$$

To specify the dynamics of functional users, one needs to consider that they are wary that the bubble created by capital gain seekers may burst causing capital losses to them. As before, bigger the bubble, greater is the chance of it bursting in the next period. Size of the bubble is proportional to the number of capital gain seekers in the market. Hence, one may argue that the probability of a functional user exiting the market in anticipation of an imminent capital loss as proportional to C_{t-1} . Introducing a constant of proportionality,, one may write the probability as hC_{t-1} . That is, the expected number of functional users exiting the market due to anticipated imminent capital losses is given by $hC_{t-1} F_{t-1}$. The dynamics of functional users can then by summarized as:

$$F_t = l(C_{t-1} + F_{t-1}) - hC_{t-1} F_{t-1} \tag{19}$$

Where l captures the strength of word of mouth influence on functional users.

To simplify matters further, one may write:

$$F_t = l(C_{t-1} + F_{t-1}) - hC_{t-1} F_{t-1} \sim qC_{t-1} \tag{20}$$

So,

$$F_{t-1} \sim qC_{t-2} \tag{21}$$

Substituting (21) in (18):

$$C_t = a(C_{t-1} + qC_{t-2}) - bC_{t-1}^2 \tag{22}$$

$$\Rightarrow C_t = aC_{t-1} + aqC_{t-2} - bC_{t-1}^2 \tag{23}$$

Carry out the following variable transformation: $x_{t-1} = \frac{b}{a} C_{t-1}$.

$$\Rightarrow x_t = ax_{t-1}(1 - x_{t-1}) + aqx_{t-2} \tag{24}$$

(24) is a delay logistic-Hénon map in which chaos arises. See Skiadas and Skiadas (2008) for details of the associated chaotic attractor.

5. DISCUSSIONS AND CONCLUSIONS

Even though the chosen dynamics are that of logistic and delay logistic-Hénon maps, the emergence of chaos is a general property of nonlinear maps. One can choose different dynamics and chaos is still likely to arise as long as the resulting map is nonlinear. The source of nonlinearity is the presence of capital gain seekers. For this reason, it seems that chaos is likely to arise in the bitcoins market.

One way to pre-empt chaos is to limit the role of purely capital gain seekers in the market. I suggest that a regulation be put in place to this effect. One possibility is to constrain online exchanges to convert bitcoins into dollars and vice versa if and only if there is an associated transaction involving goods and services or if the bitcoins are freshly mined. Even if the capital gain seekers are no longer present, the currency will still be quite volatile as the functional benefits are endogenously generated through mutual use and vary from user to user. However, if the currency gains greater acceptability, the average value of discounted expected benefits from its functional use (transaction cost savings over traditional fiat currencies) may form an anchor for expectations, enabling the possible emergence of a stable equilibrium. This may happen as the routes to chaos would be blocked as, without capital gain seekers, nonlinearity disappears. Finally, I would like to point out that this paper abstracts away from the law enforcement challenges arising from the use of bitcoins and other virtual currencies. See Trautman

(2014) for a discussion of associated law enforcement challenges.

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